Poynting vector, heating rate, and stored energy in structured materials: A first-principles derivation

M. G. Silveirinha*
Department of Electrical Engineering, Instituto de Telecomunicações, University of Coimbra, Coimbra, Portugal

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Here, we describe a first-principles derivation of the macroscopic Poynting vector, heating rate, and stored energy in arbitrary composite media formed by dielectric and metallic inclusions, taking into account the effects of artificial magnetism, bianisotropy, as well as spatial dispersion. Starting from the microscopic Maxwell's equations in an arbitrary periodic structured material, we demonstrate that in some situations it is possible to obtain a mathematically exact relation between quadratic expressions of the microscopic fields (such as the cell-averaged microscopic Poynting vector) and the macroscopic electromagnetic fields and the effective dielectric function.

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I. INTRODUCTION

Structured materials with unusual electromagnetic properties have received significant attention after several influential works[1–4] revealed that by tailoring the microstructure of conventional metals and dielectrics it is possible to change radically the propagation of light in such media. Remarkable effects, such as negative refraction,[5,6] subwavelength imaging,[7,8] cloaking,[9,10] and inversion of the palette of refracted colors by a lossless metamaterial prism,[11] have been theoretically predicted and (in some cases) experimentally demonstrated.

Somehow similar to conventional crystalline materials, metamaterials typically consist of many identical inclusions arranged in a regular lattice. The dimensions of the inclusions are much smaller than the wavelength of radiation. The study of such complex systems is much simplified by the use of homogenization techniques, which enable the characterization of the electromagnetic wave propagation using only a few *effective parameters*, in the simplest case two scalars: an effective permittivity and an effective permeability. Indeed, an important characteristic of metamaterials is that their magnetic response may be quite strong, notwithstanding the fact that the basic constituents of the material (typically metallic or dielectric particles) do not have intrinsic magnetic properties.[1] This artificial magnetism is induced by the vortex part of the electric current induced in the inclusions, which in some scenarios may mimic very closely the response of a true magnetic particle.[12]

Even though the use of homogenization methods to characterize electromagnetic wave propagation in composite media has a quite long history[13] and fundamental concepts such as "heating rate," "stored electromagnetic energy," and "Poynting vector" are well established, at least in connection to the propagation of electromagnetic waves in matter, apparently, as a survey of the recent scientific literature reveals[14–17] the definitions of such quadratic macroscopic entities is still a reason of some controversy. In particular, in Ref. 14 it was claimed the usual formulas for the macroscopic heating rate and Poynting vector are inapplicable in magnetic polarizable media and based on such conclusion it was concluded that negative refraction is impossible. Specifically, it was argued that the usual textbook formula for the macroscopic (and time-averaged) Poynting vector $\mathbf{S}_{av}=\frac{1}{2}\mathrm{Re}\{\mathbf{E}\times\mathbf{H}^*\}$ is incorrect and should be replaced by the alternative formula $\mathbf{S}_{av}=\frac{1}{2}\mathrm{Re}\{\mathbf{E}\times\frac{\mathbf{B}^*}{\mu_0}\}$ (in this work we use SI unities and assume a time harmonic variation of the type $e^{-i\omega t}$). In Ref. 16 the authors have reached a similar conclusion based on a microscopic phenomenological model. However, as it will be clearly demonstrated in this work and also addressed in part in Ref. 15, such conclusions are founded on fundamental misconceptions and mistakes.

The objective of this paper is to present a first-principles derivation of quadratic physical entities such as the Poynting vector in arbitrary structured materials. Starting from the microscopic Maxwell's equations in an arbitrary metallic-dielectric periodic material, we calculate rigorously the macroscopic expression of the Poynting vector ($\mathbf{S}_{av}$), heating rate (q_{av}), and stored electromagnetic energy (W_{av}), in terms of the macroscopic fields and of the effective dielectric function. Surprisingly, we find out that for a Bloch-Floquet mode in a lossless metamaterial it is possible to obtain a *mathematically exact* relation between the cell-averaged microscopic quadratic functions of interest (e.g., the Poynting vector) and the macroscopic electromagnetic fields. Moreover, the obtained formulas are coincident with the well-known textbook formulas of classical electrodynamics. Therefore, our analysis establishes that in some situations the traditional formulas for the macroscopic Poynting vector, heating rate, and stored energy (understood as the spatial average of the corresponding microscopic entities) are absolutely exact. Moreover, we demonstrate that in some circumstances it is also possible to calculate the exact expression of the macroscopic Poynting vector when the microscopic fields are a superposition of an arbitrary number of Bloch-Floquet natural modes, possibly associated with complex wave vectors.

Our analysis is completely general and takes into account possible effects of spatial dispersion in the composite material. Indeed, we treat the special case where the material's response can be regarded as local (described by local effective parameters, such as a local permittivity and a local permeability) as a particular case of a general material with a nonlocal response.

It is important to make clear that the results obtained in this work are based on the assumption that the microscopic (time-averaged) Poynting in a structured material formed by dielectric or metallic particles (with no magnetic properties) is $\frac{1}{2}\mathrm{Re}\{\mathbf{e}\times\frac{\mathbf{b}^*}{\mu_0}\}$, where $\mathbf{e}$ and $\mathbf{b}$ are the microscopic electric and magnetic induction fields. Indeed, even the very definition of the Poynting vector (as the flow of electromagnetic energy *at a point*) is also a subject of some debate[18,19] but such discussion is out of the scope of the present paper. Our objective is only to obtain an expression for the macroscopic Poynting vector (understood as the cell-averaged microscopic Poynting vector) consistent with the conventional definition of the microscopic Poynting vector. It is also essential to clarify the meaning of the word "microscopic" in the context of the present work. Since we are dealing with structured materials we can consider two different levels of homogenization. The first level of homogenization is the traditional one and enables the description of the interaction of electromagnetic radiation with each and every individual metallic or dielectric inclusion as if it were made of a continuous material with permittivity $\varepsilon(\mathbf{r})$. The second level of homogenization enables the description of the array of inclusions as a continuous (meta) material. By definition, here the microscopic fields and the microscopic Poynting vector are the ones defined after the first level of homogenization, whereas the macroscopic fields are the ones determined after the second level of homogenization.

This paper is organized as follows. In Sec. II, we describe the homogenization formalism considered in this work, laying the foundations for the strict mathematical analysis of the problem. In Sec. III, we present the first-principles derivation of the macroscopic Poynting vector in arbitrary metallic-dielectric metamaterials. In Sec. IV, we calculate the macroscopic stored energy and the macroscopic heating rate. Finally, in Sec. V we draw the conclusion.

II. HOMOGENIZATION FORMALISM

In this section, we describe with details the microscopic theory used to define the effective parameters of the structured material and explain the adopted constitutive relations and the averaging procedure. These concepts and ideas put on firm theoretical ground, the homogenization process, and will be used in Sec. III to relate the spatially averaged Poynting vector to the macroscopic electromagnetic fields.

We follow the method introduced in our previous work,[20,21] which enables modeling an arbitrary periodic metamaterial formed by dielectric or metallic particles using a nonlocal dielectric function of the type $\overline{\overline{\varepsilon_{eff}}}(\omega,\mathbf{k})$, where ω is the angular frequency and $\mathbf{k}=(k_x,k_y,k_z)$ is the wave vector. Such approach is valid for both materials with strong spatial dispersion and materials with weak spatial dispersion. In particular, as discussed in Refs. 20, 22, and 23, if a metamaterial can be accurately modeled with the traditional bianisotropic constitutive relations,[24,25] then it may be as well characterized by a spatially dispersive model such that the nonlocal dielectric function is linked to the local parameters as follows:

$$\frac{\overline{\overline{\varepsilon_{eff}}}}{\varepsilon_0}(\omega,\mathbf{k})=\overline{\overline{\varepsilon_r}}-\overline{\overline{\xi}}\cdot\overline{\overline{\mu_r}}^{-1}\cdot\overline{\overline{\zeta}}+\frac{c}{\omega}(\overline{\overline{\xi}}\cdot\overline{\overline{\mu_r}}^{-1}\times\mathbf{k}-\mathbf{k}\times\overline{\overline{\mu_r}}^{-1}\cdot\overline{\overline{\zeta}})$$
$$+\frac{c^2}{\omega^2}\mathbf{k}\times(\overline{\overline{\mu_r}}^{-1}-\overline{\overline{\mathbf{I}}})\times\mathbf{k}, \tag{1}$$

where $\overline{\overline{\mathbf{I}}}$ is the identity dyadic, $\overline{\overline{\varepsilon_r}}(\omega)$ is the relative *local* permittivity dyadic, $\overline{\overline{\mu_r}}(\omega)$ is the relative local permeability, and $\overline{\overline{\xi}}(\omega)$ and $\overline{\overline{\zeta}}(\omega)$ are tensors that characterize the magnetoelectric coupling. For more details the reader is referred to Ref. 20.

In the next sections we present a brief overview of the theory of Ref. 20, explaining how the nonlocal dielectric function may be determined for a general metamaterial. It should be noted that there are some notational changes as compared to Ref. 20 because in this work the microscopic fields are denoted with lowercase letters whereas the macroscopic fields are represented using capital letters and in addition here we adopt the time convention $e^{-i\omega t}$ while in Ref. 20 the time convention $e^{j\omega t}$ was assumed.

A. Spatial averaging

We consider a periodic structured material formed by metallic or dielectric inclusions of arbitrary shape, and characterized by the permittivity $\varepsilon=\varepsilon(\mathbf{r})$ and $\mu=\mu_0$ (all the materials are nonmagnetic). The unit cell of the periodic material is denoted by Ω. The Maxwell's equations in the structured material are of the form

$$\nabla\times\mathbf{e}=i\omega\mathbf{b};\quad \nabla\times\frac{\mathbf{b}}{\mu_0}=\mathbf{j}_e+\mathbf{j}_d-i\omega\varepsilon_0\mathbf{e}, \tag{2}$$

where $\mathbf{e}$ and $\mathbf{b}$ are the microscopic electric and magnetic induction fields, respectively, $\mathbf{j}_e$ is the external (applied) density of electric current, and $\mathbf{j}_d=-i\omega(\varepsilon-\varepsilon_0)\mathbf{e}$ is the density of current induced in the inclusions.

Following the ideas of Ref. 26 (see also Ref. 27), the average electric and magnetic induction fields may be defined through a spatial convolution with a suitable test function $f(\mathbf{r})$. The test function must be real valued, nonzero in some neighborhood of the origin, and such that when integrated over all space the result is unity. The support of the test function has a scale length comparable to the lattice constant. The average macroscopic electric field $\mathbf{E}$ is by definition

$$\mathbf{E}(\mathbf{r})=\int\mathbf{e}(\mathbf{r}-\mathbf{r}')f(\mathbf{r}')d^3\mathbf{r}', \tag{3}$$

and the macroscopic magnetic induction field $\mathbf{B}$ and the macroscopic external density of current $\mathbf{J}_e$ are defined similarly. The averaging procedure [Eq. (3)] has the important property that it preserves the structure of Maxwell's equations.[26] Particularly, in the homogenized medium we have that

$$\nabla\times\mathbf{E}=i\omega\mathbf{B};\quad \nabla\times\frac{\mathbf{B}}{\mu_0}=\mathbf{J}_e-i\omega(\varepsilon_0\mathbf{E}+\mathbf{P}_g), \tag{4}$$

where by definition $\mathbf{P}_g(\mathbf{r})=\frac{1}{-i\omega}\int\mathbf{j}_d(\mathbf{r}-\mathbf{r}')f(\mathbf{r}')d^3\mathbf{r}'$ is the generalized polarization vector obtained by averaging the microscopic currents, $\mathbf{j}_d=-i\omega(\varepsilon-\varepsilon_0)\mathbf{e}$, induced in the structured material. The key problem in homogenization theory is to relate $\mathbf{P}_g$ with the macroscopic electric field $\mathbf{E}$. For spatially dispersive materials the following constitutive relation is assumed:

$$\mathbf{D}_g(\mathbf{r}) = \int \overline{\overline{\hat{\varepsilon}_{eff}}}(\omega,\mathbf{r}')\mathbf{E}(\mathbf{r}-\mathbf{r}')d^3\mathbf{r}', \tag{5}$$

where $\mathbf{D}_g=\varepsilon_0\mathbf{E}+\mathbf{P}_g$ is the electric displacement and $\overline{\overline{\hat{\varepsilon}_{eff}}}(\omega,\mathbf{r})$ is the nonlocal dielectric function of the material in the space domain.

It should be noted that the averaging procedure [Eq. (3)] is equivalent to spatial filtering.[26] Specifically, the convolution operation may be regarded as low-pass filtering that eliminates the fluctuations of the microscopic fields over scale lengths comparable to the characteristic dimension of the inclusions. Based on this observation, it is interesting to consider the case where the test function corresponds to an ideal low-pass filter. Indeed, consider the test function determined by

$$\tilde{f}(\mathbf{k}) = \begin{cases} 1, & \mathbf{k} \in \mathrm{BZ} \\ 0, & \mathbf{k} \notin \mathrm{BZ} \end{cases}, \tag{6}$$

where BZ represents the first Brillouin zone and $\tilde{f}(\mathbf{k}) = \int f(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{r}$ is the Fourier transform of the test function. Moreover, consider the situation where the microscopic electric field $\mathbf{e}$ is a Bloch-Floquet wave associated with the (real-valued) wave vector $\mathbf{k}$ (assumed to lie in the first Brillouin zone). Then, it is simple to verify that the average electric field is given by

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{\mathrm{av}}e^{i\mathbf{k}\cdot\mathbf{r}}, \quad \mathbf{E}_{\mathrm{av}} = \langle\mathbf{e}\rangle, \tag{7a}$$

$$\langle\mathbf{e}\rangle \equiv \frac{1}{V_{cell}}\int_\Omega \mathbf{e}(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{r}, \tag{7b}$$

where Ω is the unit cell. Thus, the average field associated with a Bloch-Floquet wave is a plane wave whose vector amplitude is determined by the zero-order Fourier harmonic $\mathbf{E}_{\mathrm{av}}$. Due to this important property, which enables us to identify spatially averaged Bloch-Floquet waves with plane waves, in this work we will assume that the test function is determined by Eq. (6).

In these circumstances, and still considering that the microscopic fields have a Bloch-Floquet spatial variation associated with a certain wave vector $\mathbf{k}$, it is found from Eqs. (5) and (7) that $\mathbf{D}_g(\mathbf{r})=\mathbf{D}_{g,\mathrm{av}}e^{i\mathbf{k}\cdot\mathbf{r}}$ and $\mathbf{P}_g(\mathbf{r})=\mathbf{P}_{g,\mathrm{av}}e^{i\mathbf{k}\cdot\mathbf{r}}$ with

$$\mathbf{D}_{g,\mathrm{av}} = \varepsilon_0\mathbf{E}_{\mathrm{av}} + \mathbf{P}_{g,\mathrm{av}} = \overline{\overline{\varepsilon_{eff}}}(\omega,\mathbf{k}) \cdot \mathbf{E}_{\mathrm{av}}, \tag{8}$$

$$\mathbf{P}_{g,\mathrm{av}} = \frac{1}{V_{cell}}\frac{1}{-i\omega}\int_\Omega \mathbf{j}_d(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{r} \tag{9}$$

being $\overline{\overline{\varepsilon_{eff}}}(\omega,\mathbf{k})=\int\overline{\overline{\hat{\varepsilon}_{eff}}}(\omega,\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{r}$ the nonlocal dielectric function in the spectral domain. Moreover, from the macroscopic Maxwell's equations in Eq. (4), it is readily seen that the following relations are verified:[20]

$$-\mathbf{k} \times \mathbf{E}_{\mathrm{av}} + \omega\mathbf{B}_{\mathrm{av}} = 0, \tag{10a}$$

$$\omega(\varepsilon_0\mathbf{E}_{\mathrm{av}} + \mathbf{P}_{g,\mathrm{av}}) + \mathbf{k} \times \frac{\mathbf{B}_{\mathrm{av}}}{\mu_0} = -\omega\mathbf{P}_{e,\mathrm{av}}, \tag{10b}$$

where $\mathbf{P}_{e,\mathrm{av}}=\mathbf{J}_{e,\mathrm{av}}/(-i\omega)$, and $\mathbf{B}_{\mathrm{av}}$ and $\mathbf{J}_{e,\mathrm{av}}$ are the zero-order Fourier harmonics of the magnetic induction field $\mathbf{b}$ and external density of current $\mathbf{j}_e$, respectively.

B. Nonlocal dielectric function

It should be clear from Eq. (8) that to compute the nonlocal $\overline{\overline{\varepsilon_{eff}}}$ dielectric function for fixed $(\omega,\mathbf{k})$ it is sufficient to determine the generalized polarization vector $\mathbf{P}_{g,\mathrm{av}}$ associated with three independent vectors $\mathbf{E}_{\mathrm{av}}$. Based on this simple observation, in Ref. 20 we suggested that the nonlocal dielectric function may be obtained by solving the microscopic Maxwell's equations in Eq. (2) for an external current density of the form $\mathbf{j}_e=\mathbf{J}_{e,\mathrm{av}}e^{i\mathbf{k}\cdot\mathbf{r}}$, with $\mathbf{J}_{e,\mathrm{av}}$ a constant vector. Indeed, for such excitation the microscopic fields are clearly Bloch-Floquet waves associated with the wave vector $\mathbf{k}$ (imposed by the external source) and thus the corresponding average electric field may be determined in terms of the microscopic electric field using Eq. (7), whereas the generalized polarization vector is given by Eq. (9). Hence, by solving the microscopic Maxwell's equations in Eq. (2) for three independent vectors $\mathbf{J}_{e,\mathrm{av}}$ (e.g., $\mathbf{J}_{e,\mathrm{av}}\sim\hat{\mathbf{u}}_i$, $i=1,2,3$) and by computing the associated $\mathbf{E}_{\mathrm{av}}$ and $\mathbf{P}_{g,\mathrm{av}}$ in terms of the microscopic fields, it is possible to determine the unknown nonlocal dielectric function.

A useful result derived in Ref. 20, which will be instrumental in Sec. III, is that the homogenization problem may be reformulated as the following integral-differential system:

$$\nabla \times \mathbf{e} = i\omega\mathbf{b}, \tag{11a}$$

$$\nabla \times \frac{\mathbf{b}}{\mu_0} = -i\omega\varepsilon\mathbf{e} - i\omega[\hat{\mathbf{P}}_{\mathrm{av}}(\mathbf{E}_{\mathrm{av}}) - \hat{\mathbf{P}}(\mathbf{e})]e^{i\mathbf{k}\cdot\mathbf{r}} \tag{11b}$$

being $\hat{\mathbf{P}}$ and $\hat{\mathbf{P}}_{\mathrm{av}}$ the following operators (here for simplicity, and in contrast with Ref. 20, we do not consider explicitly the possibility of perfect electric conducting inclusions),

$$\hat{\mathbf{P}}(\mathbf{e}) = \frac{1}{V_{cell}}\int_\Omega [\varepsilon(\mathbf{r}) - \varepsilon_0]\mathbf{e}(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{r}, \tag{12a}$$

$$\hat{\mathbf{P}}_{\mathrm{av}}(\mathbf{E}_{\mathrm{av}}) = \frac{1}{\omega^2\mu_0}\{[k^2 - (\omega/c)^2]\overline{\overline{\mathbf{I}}} - \mathbf{k}\mathbf{k}\} \cdot \mathbf{E}_{\mathrm{av}} \tag{12b}$$

where $k^2=\mathbf{k}\cdot\mathbf{k}$ and $\mathbf{k}\mathbf{k}\equiv\mathbf{k}\otimes\mathbf{k}$ is the dyadic (tensor) product of two vectors. The operator $\hat{\mathbf{P}}$ maps a vector field into a constant vector, whereas $\hat{\mathbf{P}}_{\mathrm{av}}$ maps vectors into vectors.

In this formulation, $\mathbf{E}_{\mathrm{av}}$ should be regarded as a given parameter (i.e., the "source" or "input") or in a different perspective as the *enforced* macroscopic field ($\mathbf{E}_{\mathrm{av}}$) that determines the unknown microscopic fields $(\mathbf{e},\mathbf{b})$. In this framework the external current, $\mathbf{j}_e=\mathbf{J}_{e,\mathrm{av}}e^{i\mathbf{k}\cdot\mathbf{r}}$, satisfies $\mathbf{J}_{e,\mathrm{av}} = -i\omega[\hat{\mathbf{P}}_{\mathrm{av}}(\mathbf{E}_{\mathrm{av}})-\hat{\mathbf{P}}(\mathbf{e})]$, i.e., depends explicitly on the enforced macroscopic electric field and on the unknown microscopic electric field. The dielectric function $\overline{\overline{\varepsilon_{eff}}}(\omega,\mathbf{k})$ can be computed by solving the integral-differential system [Eq.

(11)] for three independent vectors $\mathbf{E}_{\rm av}$ (e.g., $\mathbf{E}_{\rm av}\sim\hat{\mathbf{u}}_i$, $i=1,2,3$) and by computing the associated $\mathbf{P}_{g,\rm av}$ in terms of the microscopic fields.

In Ref. 20 it was proven that the solution $(\mathbf{e},\mathbf{b})$ of the homogenization problem varies smoothly with the parameters $(\omega,\mathbf{k},\mathbf{E}_{\rm av})$, even in the vicinity of points associated with a Bloch-Floquet natural mode of the material [solution of Eq. (2) with $\mathbf{j}_e=0$]. Clearly, if $(\omega,\mathbf{k},\mathbf{E}_{\rm av})$ is associated with a Bloch-Floquet natural mode then the corresponding solution of the system [Eq. (11)] is such that the external current, $\mathbf{J}_{e,\rm av}=-i\omega[\hat{\mathbf{P}}_{\rm av}(\mathbf{E}_{\rm av})-\hat{\mathbf{P}}(\mathbf{e})]$, vanishes. For further details the reader is referred to Ref. 20.

III. MACROSCOPIC POYNTING VECTOR

Here, we will show that when the metamaterial is lossless and the microscopic field is a Bloch-Floquet wave it is possible to obtain an exact mathematical relation between the macroscopic Poynting vector, the macroscopic fields and the nonlocal dielectric function. In particular, we will prove that for a local metamaterial with a magnetic response the macroscopic Poynting vector reduces to the classical textbook formula.

In Sec. III A, we obtain a fundamental mathematical result that establishes the relation between the spatial average of an auxiliary quadratic expression involving the microscopic fields and the macroscopic fields. In Sec. III B, we apply that result to calculate the macroscopic Poynting vector associated with a Bloch-Floquet natural mode. Finally, in Sec. III C it is shown that in some scenarios it is also possible to compute the macroscopic Poynting vector associated with a superposition of Bloch-Floquet modes in closed analytical form.

Since by hypothesis the metamaterial is lossless and periodic, it is assumed here that the permittivity of the inclusions, $\varepsilon=\varepsilon(\mathbf{r})$, is periodic and real valued.

A. Spatial average of the product of Bloch modes

Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be Bloch-Floquet natural modes of the periodic material (associated with the wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively). Thus, $\mathbf{e}_1$ and $\mathbf{e}_2$ satisfy the microscopic Maxwell's equations in Eq. (2) with $\mathbf{j}_e=0$ or equivalently

$$\nabla\times\nabla\times\mathbf{e}=\omega^2\varepsilon\mu_0\mathbf{e}. \tag{13}$$

It is assumed that the wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, which in general may be complex valued, are linked by the relation (the "$*$" denotes complex conjugation),

$$\mathbf{k}_1=\mathbf{k}_2^*. \tag{14}$$

The above condition ensures that $\mathbf{e}_1$ and $\mathbf{e}_2^*$ are Bloch-Floquet waves associated with symmetric wave vectors, and, in particular, $\mathbf{e}_1\cdot\mathbf{e}_2^*$ and $\mathbf{e}_1\times\mathbf{e}_2^*$ are periodic functions. In the following it is formally demonstrated that the vector $\mathbf{s}_{1,2}$ defined by

$$\mathbf{s}_{1,2}\equiv\frac{1}{-4i\omega\mu_0}(\mathbf{e}_1\times\nabla\times\mathbf{e}_2^*-\mathbf{e}_2^*\times\nabla\times\mathbf{e}_1) \tag{15}$$

verifies

$$(\mathbf{s}_{1,2})_{\mathrm{av},l}=\frac{1}{4}\left(\mathbf{E}_{\mathrm{av},1}\times\frac{\mathbf{B}_{\mathrm{av},2}^*}{\mu_0}+\mathbf{E}_{\mathrm{av},2}^*\times\frac{\mathbf{B}_{\mathrm{av},1}}{\mu_0}\right)_l - \frac{\omega}{4}\mathbf{E}_{\mathrm{av},2}^*\cdot\frac{\partial\overline{\overline{\varepsilon_{eff}}}}{\partial k_l}(\omega,\mathbf{k}_1)\cdot\mathbf{E}_{\mathrm{av},1}\quad (l=x,y,z), \tag{16}$$

where $\overline{\overline{\varepsilon_{eff}}}$ is the dielectric function of the composite material defined as in Sec. II, $\mathbf{E}_{\mathrm{av},n}$ and $\mathbf{B}_{\mathrm{av},n}$ are the macroscopic averaged electric and magnetic induction fields associated with the microscopic field $\mathbf{e}_n$ $(n=1,2)$, defined consistently with Eq. (7) and linked by Eq. (10a), and $(\mathbf{s}_{1,2})_{\rm av}$ is the spatial average of $\mathbf{s}_{1,2}$ in the unit cell,

$$(\mathbf{s}_{1,2})_{\rm av}=\frac{1}{V_{cell}}\int_\Omega \mathbf{s}_{1,2}d^3\mathbf{r}. \tag{17}$$

In Eq. (16) the subscript l refers to the lth Cartesian component of a given vector [e.g., $(\mathbf{s}_{1,2})_{\mathrm{av},l}\equiv(\mathbf{s}_{1,2})_{\rm av}\cdot\hat{\mathbf{u}}_l$]. It should be clear that because of Eq. (14) $\mathbf{s}_{1,2}$ is a periodic function in the lattice.

In order to demonstrate the enunciated result, first we note that $\mathbf{s}_{1,2}$ may be rewritten as

$$(\mathbf{s}_{1,2})_l=\frac{1}{4\omega\mu_0}[\mathbf{e}_1\cdot\nabla\times\nabla\times(ix_l\mathbf{e}_2)^*-(ix_l\mathbf{e}_2)^*\cdot\nabla\times\nabla\times\mathbf{e}_1 + i\nabla\cdot\{(\hat{\mathbf{u}}_l\times\mathbf{e}_2^*)\times\mathbf{e}_1\}+ix_l(\mathbf{e}_1\cdot\nabla\times\nabla\times\mathbf{e}_2^*-\mathbf{e}_2^*\cdot\nabla\times\nabla\times\mathbf{e}_1)]. \tag{18}$$

The last term in the right-hand side of the above equation vanishes because the fields $\mathbf{e}_1$ and $\mathbf{e}_2$ satisfy Eq. (13), and the material is assumed lossless [$\varepsilon=\varepsilon(\mathbf{r})$ is real valued].

On the other hand, if $\mathbf{f}_2$ is an arbitrary vector field the following vector identity holds:

$$0=\mathbf{e}_1\cdot\nabla\times\nabla\times\mathbf{f}_2^*-\mathbf{f}_2^*\cdot\nabla\times\nabla\times\mathbf{e}_1 -\nabla\cdot(\mathbf{f}_2^*\times\nabla\times\mathbf{e}_1-\mathbf{e}_1\times\nabla\times\mathbf{f}_2^*). \tag{19}$$

Let us assume that $\mathbf{f}_2$ also verifies the Bloch-Floquet property and is associated with the same wave vector ($\mathbf{k}_2$) as $\mathbf{e}_2$ (however, $\mathbf{f}_2$ is not necessarily a solution of Maxwell's equations). Then, adding Eqs. (18) and (19) member by member and integrating the resulting equation over the unit cell, and using Eq. (13), it is found that

$$(\mathbf{s}_{1,2})_{\mathrm{av},l}=\frac{1}{V_{cell}}\frac{1}{4\omega\mu_0}\int_\Omega \mathbf{e}_1\cdot[\nabla\times\nabla\times(ix_l\mathbf{e}_2+\mathbf{f}_2) -\omega^2\varepsilon\mu_0(ix_l\mathbf{e}_2+\mathbf{f}_2)]^*d^3\mathbf{r}. \tag{20}$$

To obtain the above result we took into account that the volume integrals of terms of the type $\nabla\cdot\{\cdots\}$ vanish because of Gauss's theorem and of the assumed Bloch-Floquet conditions.

To proceed, we consider the family of solutions of the homogenization problem [Eq. (11)] parameterized by the wave vector $\mathbf{k}$, $\mathbf{e}_{2,F}=\mathbf{e}_{2,F}(\mathbf{r};\mathbf{k})$, with the parameter $\mathbf{E}_{\rm av}$ in Eq. (11b) equal to $\mathbf{E}_{\mathrm{av},2}=\langle\mathbf{e}_2\rangle$. It should be clear that the spatial average of $\mathbf{e}_{2,F}$, defined as in Eq. (7b), verifies $\langle\mathbf{e}_{2,F}\rangle=\mathbf{E}_{\mathrm{av},2}$, independent of $\mathbf{k}$. In other words the family $\mathbf{e}_{2,F}=\mathbf{e}_{2,F}(\mathbf{r};\mathbf{k})$

yields after proper homogenization the family of plane waves $\mathbf{E}_{\mathrm{av},2}e^{i\mathbf{k}\cdot\mathbf{r}}$. Clearly, $\mathbf{e}_{2,F}$ verifies the integral-differential system,

$$\nabla\times\nabla\times\mathbf{e}_{2,F}=+i\omega\mu_0\mathbf{J}_{\mathrm{av},2}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}+\omega^2\varepsilon\mu_0\mathbf{e}_{2,F}, \quad (21)$$

where the amplitude of the external source verifies

$$\mathbf{J}_{\mathrm{av},2}(\mathbf{k})=-i\omega[\hat{\mathbf{P}}_{\mathrm{av}}(\mathbf{E}_{\mathrm{av},2})-\hat{\mathbf{P}}(\mathbf{e}_{2,F})]. \quad (22)$$

The solution of the system [Eq. (21)] is a smooth function of the wave vector. Moreover, since $\langle\mathbf{e}_2\rangle=\mathbf{E}_{\mathrm{av},2}$ it should be clear from the discussion in the end of Sec. II that for $\mathbf{k}=\mathbf{k}_2$ the amplitude of the external density of current given by Eq. (22) vanishes: $\mathbf{J}_{\mathrm{av},2}(\mathbf{k}_2)=0$. Hence, from Eq. (13) it follows that $\mathbf{e}_2$ is a solution of the homogenization problem [Eq. (21)] associated with $\mathbf{k}=\mathbf{k}_2$, i.e., we have proven that $\mathbf{e}_2=\mathbf{e}_{2,F}(\mathbf{r};\mathbf{k}=\mathbf{k}_2)$. Therefore, we may drop the subscript "F" in the definition of $\mathbf{e}_{2,F}$ and regard the original electromagnetic field $\mathbf{e}_2$ as a function of both $\mathbf{r}$ and $\mathbf{k}$, i.e., $\mathbf{e}_2=\mathbf{e}_2(\mathbf{r};\mathbf{k})$.

In particular, calculating the derivative of both members of Eq. (21) with respect to k_l at $\mathbf{k}=\mathbf{k}_2$, it is found that

$$\nabla\times\nabla\times\frac{\partial\mathbf{e}_2}{\partial k_l}=+i\omega\mu_0\frac{\partial\mathbf{J}_{\mathrm{av},2}}{\partial k_l}e^{i\mathbf{k}_2\cdot\mathbf{r}}+\omega^2\varepsilon\mu_0\frac{\partial\mathbf{e}_2}{\partial k_l},$$

$$\mathbf{k}=\mathbf{k}_2 \quad (l=x,y,z). \quad (23)$$

Next we note that since $\mathbf{e}_2$ is a Bloch-Floquet wave [i.e., $\mathbf{e}_2(\mathbf{r};\mathbf{k})=\mathbf{u}(\mathbf{r};\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}$ with $\mathbf{u}$ a periodic function] we have that $\frac{\partial\mathbf{e}_2}{\partial k_l}$ is of the form $\frac{\partial\mathbf{e}_2}{\partial k_l}=ix_l\mathbf{e}_2+\mathbf{f}_2$, being $\mathbf{f}_2$ a Bloch-Floquet wave also associated with $\mathbf{k}_2$. Hence, we may replace $ix_l\mathbf{e}_2+\mathbf{f}_2$ by $\frac{\partial\mathbf{e}_2}{\partial k_l}$ in Eq. (20). Using Eq. (23), it is seen that

$$(\mathbf{s}_{1,2})_{\mathrm{av},l}=\frac{1}{V_{cell}}\frac{-i}{4}\int_\Omega\mathbf{e}_1(\mathbf{r})\cdot\frac{\partial\mathbf{J}^*_{\mathrm{av},2}}{\partial k_l}e^{-i\mathbf{k}_2^*\cdot\mathbf{r}}d^3\mathbf{r}. \quad (24)$$

From the definition [Eq. (22)] it should be clear that $\mathbf{J}_{\mathrm{av},2}$ is independent of the space coordinates. Thus, using Eqs. (7) and (14) we find that

$$(\mathbf{s}_{1,2})_{\mathrm{av},l}=\frac{-i}{4}\mathbf{E}_{\mathrm{av},1}\cdot\left.\frac{\partial\mathbf{J}^*_{\mathrm{av},2}}{\partial k_l}\right|_{\mathbf{k}_2}. \quad (25)$$

To further simplify this result, we use the fact that $\mathbf{J}_{\mathrm{av},2}$ may be written explicitly in terms of the nonlocal dielectric function $\overline{\overline{\varepsilon_{eff}}}(\omega,\mathbf{k})$ of the composite material. Indeed, from Eqs. (8), (9), and (12), we have that

$$\mathbf{J}_{\mathrm{av},2}(\mathbf{k})=-i\omega\varepsilon_0\left[-\frac{1}{\varepsilon_0}\overline{\overline{\varepsilon_{eff}}}(\omega,\mathbf{k})+\frac{c^2}{\omega^2}k^2\overline{\overline{\mathbf{I}}}-\frac{c^2}{\omega^2}\mathbf{k}\mathbf{k}\right]\cdot\mathbf{E}_{\mathrm{av},2}. \quad (26)$$

Thus, calculating the derivative of $\mathbf{J}_{\mathrm{av},2}$ with respect to k_l and substituting in Eq. (25), we have that

$$(\mathbf{s}_{1,2})_{\mathrm{av},l}=\frac{\omega\varepsilon_0}{4}\mathbf{E}_{\mathrm{av},1}\cdot\left[-\frac{1}{\varepsilon_0}\frac{\partial\overline{\overline{\varepsilon_{eff}}}}{\partial k_l}(\omega,\mathbf{k}_2)+\frac{c^2}{\omega^2}2\hat{\mathbf{u}}_l\cdot\mathbf{k}_2\overline{\overline{\mathbf{I}}}-\frac{c^2}{\omega^2}(\hat{\mathbf{u}}_l\mathbf{k}_2+\mathbf{k}_2\hat{\mathbf{u}}_l)\right]^*\cdot\mathbf{E}^*_{\mathrm{av},2}, \quad (27)$$

where $\hat{\mathbf{u}}_l\mathbf{k}_2\equiv\hat{\mathbf{u}}_l\otimes\mathbf{k}_2$ represents the dyadic (tensor) product of two vectors. But straightforward calculations show that (using $\mathbf{k}_1=\mathbf{k}_2^*$)

$$\frac{1}{4}\left(\mathbf{E}_{\mathrm{av},1}\times\frac{\mathbf{B}^*_{\mathrm{av},2}}{\mu_0}+\mathbf{E}^*_{\mathrm{av},2}\times\frac{\mathbf{B}_{\mathrm{av},1}}{\mu_0}\right)\cdot\hat{\mathbf{u}}_l$$
$$=\frac{1}{4\omega\mu_0}\mathbf{E}_{\mathrm{av},1}\cdot[2\hat{\mathbf{u}}_l\,.\,\mathbf{k}_2\overline{\overline{\mathbf{I}}}-(\hat{\mathbf{u}}_l\mathbf{k}_2+\mathbf{k}_2\hat{\mathbf{u}}_l)]^*\cdot\mathbf{E}^*_{\mathrm{av},2}. \quad (28)$$

Thus, Eq. (27) may be rewritten as

$$(\mathbf{s}_{1,2})_{\mathrm{av},l}=\frac{1}{4}\left(\mathbf{E}_{\mathrm{av},1}\times\frac{\mathbf{B}^*_{\mathrm{av},2}}{\mu_0}+\mathbf{E}^*_{\mathrm{av},2}\times\frac{\mathbf{B}_{\mathrm{av},1}}{\mu_0}\right)_l-\frac{\omega}{4}\mathbf{E}_{\mathrm{av},1}\cdot\left[\frac{\partial\overline{\overline{\varepsilon_{eff}}}}{\partial k_l}(\omega,\mathbf{k}_2)\cdot\mathbf{E}_{\mathrm{av},2}\right]^*. \quad (29)$$

The dielectric function of a structured material verifies (ω is assumed real valued in this work; the material is reciprocal),[20,28]

$$[\overline{\overline{\varepsilon_{eff}}}(\omega,\mathbf{k})]^*=\overline{\overline{\varepsilon_{eff}}}(-\omega,-\mathbf{k}^*)=[\overline{\overline{\varepsilon_{eff}}}(-\omega,\mathbf{k}^*)]^T, \quad (30)$$

where the superscript "T" represents the transpose dyadic. Moreover, in case of a lossless material the dielectric function must be an even function of ω (this follows directly from the definition of $\overline{\overline{\varepsilon_{eff}}}$ given in Sec. II),

$$\overline{\overline{\varepsilon_{eff}}}(\omega,\mathbf{k})=\overline{\overline{\varepsilon_{eff}}}(-\omega,\mathbf{k}) \quad (\text{lossless material}). \quad (31)$$

Hence we have that

$$[\overline{\overline{\varepsilon_{eff}}}(\omega,\mathbf{k})]^*=[\overline{\overline{\varepsilon_{eff}}}(\omega,\mathbf{k}^*)]^T. \quad (32)$$

Substituting the above formula into Eq. (29), and using again the relation $\mathbf{k}_1=\mathbf{k}_2^*$, we finally obtain the desired relation [Eq. (16)].

To conclude this section, we mention that by slightly modifying the previous demonstration, it can be shown that Eq. (16) remains valid even when $\mathbf{e}_1$ and $\mathbf{e}_2$ are not natural modes of the material. The only essential conditions are that Eq. (14) holds and that $\mathbf{e}_1$ and $\mathbf{e}_2$ are Bloch-Floquet waves that verify the microscopic Maxwell's equations in Eq. (2) for a source of the type $\mathbf{j}_e=\mathbf{J}_{e,\mathrm{av}}e^{i\mathbf{k}\cdot\mathbf{r}}$, with $\mathbf{J}_{e,\mathrm{av}}$ a constant vector.

B. Poynting vector for a propagating mode

Using the fundamental result derived in Sec. III A, it is straightforward to calculate the macroscopic Poynting vector associated with a propagating Bloch-Floquet natural mode $\mathbf{e}=\mathbf{e}(\mathbf{r})$ of the periodic material. Indeed, when $\mathbf{k}$ is real valued we can choose $\mathbf{e}_2=\mathbf{e}_1=\mathbf{e}$ in Eq. (15). Moreover, in that case it is clear that $\mathbf{s}_{1,2}$ is coincident with the microscopic Poynting vector $\mathbf{s}=\frac{1}{2}\mathrm{Re}\{\mathbf{e}\times\frac{\mathbf{b}^*}{\mu_0}\}$. Thus, Eq. (16) establishes the following important relation:

$$\mathbf{S}_{\mathrm{av},l}=\frac{1}{2}\mathrm{Re}\left\{\left(\mathbf{E}_{\mathrm{av}}\times\frac{\mathbf{B}^*_{\mathrm{av}}}{\mu_0}\right)_l\right\}-\frac{\omega}{4}\mathbf{E}^*_{\mathrm{av}}\cdot\frac{\partial\overline{\overline{\varepsilon_{eff}}}}{\partial k_l}(\omega,\mathbf{k})\cdot\mathbf{E}_{\mathrm{av}}$$

$$(l=x,y,z), \quad (33)$$

where

$$\mathbf{S}_{\mathrm{av}} = \frac{1}{V_{cell}}\int_{\Omega} \frac{1}{2}\mathrm{Re}\left\{\mathbf{e}\times\frac{\mathbf{b}^*}{\mu_0}\right\} d^3\mathbf{r}. \tag{34}$$

It is stressed that Eq. (33) is mathematically *exact*, and only assumes that the structured material is lossless and periodic, that $\mathbf{k}$ is real valued, and that the macroscopic electromagnetic fields and the effective dielectric function are defined as in Sec. II. Quite interestingly, the derived formula is completely consistent with the well-known formula for the Poynting vector in spatially dispersive materials, reported, for example, in Refs. 28 and 29. However, the derivation reported in Ref. 28 is based solely on the macroscopic (homogenized) Maxwell's equations while the proof reported here follows directly from first principles, i.e., from microscopic theory. Notice that, as follows from Eq. (32)—see also Ref. 28—for a lossless reciprocal material and real valued $\mathbf{k}$ the nonlocal dielectric function is Hermitian, and thus the second term in the right-hand side of Eq. (33) is real valued, as it should.

Let us consider now the important particular case where the composite material is characterized by weak spatial dispersion and is such that its response is effectively local and described in terms of local permittivity and permeability tensors, $\overline{\overline{\varepsilon_r}}(\omega)$ and $\overline{\overline{\mu_r}}(\omega)$, and in addition, for gyrotropic materials, with tensors $\bar{\bar{\xi}}(\omega)$ and $\bar{\bar{\zeta}}(\omega)$ related to the magnetoelectric coupling (see Ref. 20; a somehow related analysis is presented in Ref. 30). As discussed in Sec. II, such materials can be as well characterized by a nonlocal dielectric function defined as in Eq. (1). Hence, in terms of the local parameters we have that

$$\begin{aligned}\mathbf{E}_{\mathrm{av}}^* \cdot \frac{1}{\varepsilon_0}\frac{\partial \overline{\overline{\varepsilon_{eff}}}}{\partial k_l}(\omega,\mathbf{k})\cdot\mathbf{E}_{\mathrm{av}} &= \mathbf{E}_{\mathrm{av}}^*\cdot\frac{c}{\omega}(\bar{\bar{\xi}}\cdot\overline{\overline{\mu_r}}^{-1}\times\hat{\mathbf{u}}_l - \hat{\mathbf{u}}_l\times\overline{\overline{\mu_r}}^{-1}\cdot\bar{\bar{\zeta}})\cdot\mathbf{E}_{\mathrm{av}} \\ &\quad + \mathbf{E}_{\mathrm{av}}^*\cdot\left[\frac{c^2}{\omega^2}\hat{\mathbf{u}}_l\times(\overline{\overline{\mu_r}}^{-1}-\bar{\bar{\mathbf{I}}})\times\mathbf{k} \right.\\ &\quad \left. + \frac{c^2}{\omega^2}\mathbf{k}\times(\overline{\overline{\mu_r}}^{-1}-\bar{\bar{\mathbf{I}}})\times\hat{\mathbf{u}}_l\right]\cdot\mathbf{E}_{\mathrm{av}}.\end{aligned} \tag{35}$$

Using the vector properties,

$$(\bar{\bar{\mathbf{A}}}\times\mathbf{a})\cdot\mathbf{b} = \bar{\bar{\mathbf{A}}}\cdot(\mathbf{a}\times\mathbf{b});\quad \mathbf{a}\cdot(\mathbf{b}\times\bar{\bar{\mathbf{A}}}) = (\mathbf{a}\times\mathbf{b})\cdot\bar{\bar{\mathbf{A}}} \tag{36}$$

valid for generic vectors $\mathbf{a}$ and $\mathbf{b}$ and a generic dyadic $\bar{\bar{\mathbf{A}}}$, and using the identity $\mathbf{B}_{\mathrm{av}}=\mathbf{k}\times\mathbf{E}_{\mathrm{av}}/\omega$, which follows from Eq. (10a), it is readily found that

$$\begin{aligned}\mathbf{E}_{\mathrm{av}}^*\cdot\frac{1}{\varepsilon_0}\frac{\partial\overline{\overline{\varepsilon_{eff}}}}{\partial k_l}(\omega,\mathbf{k})\cdot\mathbf{E}_{\mathrm{av}} &= \frac{c^2}{\omega}\left[\frac{1}{c}\mathbf{E}_{\mathrm{av}}^*\cdot\bar{\bar{\xi}}\cdot\overline{\overline{\mu_r}}^{-1} - \mathbf{B}_{\mathrm{av}}^*\cdot(\overline{\overline{\mu_r}}^{-1}-\bar{\bar{\mathbf{I}}})\right]\cdot(\hat{\mathbf{u}}_l\times\mathbf{E}_{\mathrm{av}}) \\ &\quad + \frac{c^2}{\omega}(\mathbf{E}_{\mathrm{av}}^*\times\hat{\mathbf{u}}_l)\cdot\left[-\frac{1}{c}\overline{\overline{\mu_r}}^{-1}\cdot\bar{\bar{\zeta}}\cdot\mathbf{E}_{\mathrm{av}} + (\overline{\overline{\mu_r}}^{-1}-\bar{\bar{\mathbf{I}}})\cdot\mathbf{B}_{\mathrm{av}}\right].\end{aligned} \tag{37}$$

Since from reciprocity $\overline{\overline{\mu_r}}$ must be symmetric and $\bar{\bar{\xi}}=-\bar{\bar{\zeta}}^T$,[24,25] and because in the lossless case $\overline{\overline{\mu_r}}$ is real valued whereas $\bar{\bar{\zeta}}$ is imaginary pure, it follows that

$$\begin{aligned}\mathbf{E}_{\mathrm{av}}^*\cdot\frac{1}{\varepsilon_0}\frac{\partial\overline{\overline{\varepsilon_{eff}}}}{\partial k_l}(\omega,\mathbf{k})\cdot\mathbf{E}_{\mathrm{av}} &= -\frac{c^2}{\omega}(\hat{\mathbf{u}}_l\times\mathbf{E}_{\mathrm{av}})\cdot\left[-\frac{1}{c}\overline{\overline{\mu_r}}^{-1}\cdot\bar{\bar{\zeta}}\cdot\mathbf{E}_{\mathrm{av}} + (\overline{\overline{\mu_r}}^{-1}-\bar{\bar{\mathbf{I}}})\cdot\mathbf{B}_{\mathrm{av}}\right]^* \\ &\quad - \frac{c^2}{\omega}(\hat{\mathbf{u}}_l\times\mathbf{E}_{\mathrm{av}}^*)\cdot\left[-\frac{1}{c}\overline{\overline{\mu_r}}^{-1}\cdot\bar{\bar{\zeta}}\cdot\mathbf{E}_{\mathrm{av}} + (\overline{\overline{\mu_r}}^{-1}-\bar{\bar{\mathbf{I}}})\cdot\mathbf{B}_{\mathrm{av}}\right].\end{aligned} \tag{38}$$

Hence, substituting the above formula into Eq. (33) and using again the identity $\mathbf{a}\cdot(\mathbf{b}\times\bar{\bar{\mathbf{A}}})=(\mathbf{a}\times\mathbf{b})\cdot\bar{\bar{\mathbf{A}}}$, we finally obtain that

$$\mathbf{S}_{\mathrm{av}} = \frac{1}{2}\mathrm{Re}\left\{\mathbf{E}_{\mathrm{av}}\times\left[\mu_0^{-1}\overline{\overline{\mu_r}}^{-1}\cdot\mathbf{B}_{\mathrm{av}} - \frac{\mu_0^{-1}}{c}\overline{\overline{\mu_r}}^{-1}\cdot\bar{\bar{\zeta}}\cdot\mathbf{E}_{\mathrm{av}}\right]^*\right\}. \tag{39}$$

The above formula gives the macroscopic Poynting vector associated with a generic local material characterized by the relative permeability tensor $\overline{\overline{\mu_r}}$ and by the magnetoelectric coupling tensor $\bar{\bar{\zeta}}$. Notably, the term in rectangular brackets in Eq. (39) is nothing more than the traditional formula for the macroscopic magnetic field in local media (described by the usual bianisotropic constitutive relations; see Refs. 20, 24, and 25): $\mathbf{H}_{\mathrm{av}}\equiv\mu_0^{-1}\overline{\overline{\mu_r}}^{-1}\cdot\mathbf{B}_{\mathrm{av}}-\frac{\mu_0^{-1}}{c}\overline{\overline{\mu_r}}^{-1}\cdot\bar{\bar{\zeta}}\cdot\mathbf{E}_{\mathrm{av}}$. Thus, our results demonstrate that for local media the correct formula for the macroscopic Poynting vector is coincident with the traditional one, i.e., $\mathbf{S}_{\mathrm{av}}=\frac{1}{2}\mathrm{Re}\{\mathbf{E}_{\mathrm{av}}\times\mathbf{H}_{\mathrm{av}}\}$, and that the claims of Refs. 14 and 16 that such formula does not apply for media with artificial magnetism are completely unsubstantiated.

Let us make some additional considerations about the results of Refs. 14 and 16. The main reason for some of the misunderstandings of Refs. 14 and 16 is the fact that the relation between the spatially averaged macroscopic fields and spatially averaged quadratic expressions like the Poynting vector is not trivial. Indeed, let $\langle . \rangle_{\mathrm{space}}$ be some (linear) operator that represents spatial averaging (not necessarily the one we adopted in Sec. II) so that $\mathbf{E}=\langle\mathbf{e}\rangle_{\mathrm{space}}$ and $\mathbf{B}=\langle\mathbf{b}\rangle_{\mathrm{space}}$. It is quite obvious that in general $\mathbf{E}\times\mathbf{B}^*\neq\langle\mathbf{e}\times\mathbf{b}^*\rangle_{\mathrm{space}}$ (in the same manner as $|\mathbf{E}|^2\neq\langle|\mathbf{e}|^2\rangle_{\mathrm{space}}$) and thus the relation between the macroscopic Poynting vector $\mathbf{S}_{\mathrm{av}}$ (which may identified with $\frac{1}{2}\mathrm{Re}\{\langle\mathbf{e}\times\mathbf{b}^*/\mu_0\rangle_{\mathrm{space}}\}$) and the macroscopic fields is not evident. The authors of Ref. 16 failed to understand this property and this lead them to the erroneous conclusion that $\mathbf{S}_{\mathrm{av}}=\frac{1}{2}\mathrm{Re}\{\mathbf{E}\times\frac{\mathbf{B}^*}{\mu_0}\}$. Similarly, in Ref. 14 [Eq. (25)] it was claimed that if we put $\mathbf{e}=\mathbf{E}+\delta\mathbf{e}$ and $\mathbf{b}=\mathbf{B}+\delta\mathbf{b}$, where $\delta\mathbf{e}$ and $\delta\mathbf{b}$ are the fluctuating parts of the microscopic fields, then $\langle\mathbf{e}\times\mathbf{b}\rangle_{\mathrm{space}}=\mathbf{E}\times\mathbf{B}$ because the term $\langle\delta\mathbf{e}\times\delta\mathbf{b}\rangle_{\mathrm{space}}$ is "quadratic in field fluctuations and can be omitted as small." However, we find such claim completely unjustified and in fact, as it is clear from our derivation, the term $\langle\delta\mathbf{e}\times\delta\mathbf{b}\rangle_{\mathrm{space}}$ cannot be neglected in the general case. It is important to emphasize that the formula $\mathbf{S}_{\text{av}}=\frac{1}{2\mu_0}\text{Re}\{\mathbf{E}\times\mathbf{B}^*\}$ could be correct only if $\mathbf{E}\times\mathbf{B}^*=\langle\mathbf{e}\times\mathbf{b}^*\rangle_{\text{space}}$, which in general is obviously false. This argument alone would be sufficient to the outright rejection of the theories of Refs. 14 and 16 (similar arguments may be used to reject the alternative formula for the heating rate suggested in Ref. 14).

C. Poynting vector for a superposition of natural modes

In Sec. III B we have established a rigorous connection between the cell-averaged microscopic Poynting vector associated with a Bloch-Floquet natural mode and the macroscopic fields and effective parameters of the homogenized medium. In order to obtain such elegant result we had to require that the material is lossless and that the wave vector is real valued. However, it would be certainly interesting to extend such result at least to the case where the electromagnetic field is a superposition of several natural modes, possibly associated with complex-valued wave vectors. In the following, we demonstrate that this is possible in some situations. Specifically, we will prove that when the microscopic electromagnetic fields are a superposition of natural modes associated with the wave vectors $\mathbf{k}_n$ $(n=1,2,\ldots)$, such that the projection of all the $\mathbf{k}_n$ onto a given plane (let us say for definiteness the xoy plane) is the same and real valued, then it is possible to write the z component of the averaged Poynting vector in terms of the macroscopic fields and of the nonlocal dielectric function. Notice that the described scenario is quite important in practical problems. For example, it is well known that when a plane wave illuminates a periodic structure (let us say a metamaterial slab), the field inside the slab can be written exclusively in terms of Bloch-Floquet modes, in general, associated with complex-valued wave vectors $\mathbf{k}_n$, and such that the projection of $\mathbf{k}_n$ onto the interface is equal to the projection of the wave vector of the incident wave onto the interface.[31]

To demonstrate the enunciated property, first we need to obtain an auxiliary result. As in Sec. III A, the metamaterial is assumed lossless. Without loss of generality, it is supposed that the unit cell of the metamaterial can be decomposed as follows:

$$\Omega=\Omega_T\times\left[-\frac{a_\perp}{2},\frac{a_\perp}{2}\right], \tag{40}$$

where Ω_T is the transverse unit cell that defines the periodicity in the xoy plane and $a_\perp$ is the lattice constant along the z direction. For convenience we introduce the following 2-form:

$$\chi(\mathbf{e}_1,\mathbf{e}_2)\equiv\frac{1}{4}\int_{\Omega_T}(\mathbf{e}_2\times\nabla\times\mathbf{e}_1^*-\mathbf{e}_1^*\times\nabla\times\mathbf{e}_2)\cdot\hat{\mathbf{u}}_z dxdy. \tag{41}$$

Clearly, $\chi(\mathbf{e},\cdot)$ is linear in the second argument and $\chi(\mathbf{e}_1,\mathbf{e}_2)=-\chi(\mathbf{e}_2,\mathbf{e}_1)^*$. Moreover, it is simple to verify that if $\mathbf{e}$ is the electric field associated with an electromagnetic wave then $\chi(\mathbf{e},\mathbf{e})/(-i\omega\mu_0)$ is the power that crosses the transverse cell in the z direction.

In general, $\chi(\mathbf{e}_1,\mathbf{e}_2)$ is a function of the z. However, when both $\mathbf{e}_1$ and $\mathbf{e}_2$ satisfy the homogeneous Maxwell's equations in a lossless material and if in addition they have the Bloch-Floquet property in the x and y coordinates (being associated with the real-valued transverse wave number $\mathbf{k}_\parallel=(k_x,k_y,0)$; the variation along z may be arbitrary), then $\chi(\mathbf{e}_1,\mathbf{e}_2)$ becomes independent of z. Indeed, from Eq. (13) it follows that $\nabla\cdot(\mathbf{e}_2\times\nabla\times\mathbf{e}_1^*-\mathbf{e}_1^*\times\nabla\times\mathbf{e}_2)=0$ and thus, applying Gauss's theorem and using the Bloch-Floquet conditions to show that the integral over the lateral wall's vanishes, it is readily found that $\chi(\mathbf{e}_1,\mathbf{e}_2)$ has the same value at two arbitrary values of z.

Another interesting property, still assuming that $\mathbf{e}_1$ and $\mathbf{e}_2$ satisfy the homogeneous Maxwell's equations and have the Bloch property in the xoy plane, is that $\chi(\mathbf{T}_{a_\perp}\mathbf{e}_1,\mathbf{T}_{a_\perp}\mathbf{e}_2)=\chi(\mathbf{e}_1,\mathbf{e}_2)$, where $\mathbf{T}_{a_\perp}$ represents the translation operator defined by $(\mathbf{T}_{a_\perp}\mathbf{e})(\mathbf{r})\equiv\mathbf{e}(\mathbf{r}+a_\perp\hat{\mathbf{u}}_z)$ for a generic field $\mathbf{e}$. In fact, from the definition [Eq. (41)] it is easy to verify that $\chi(\mathbf{T}_{a_\perp}\mathbf{e}_1,\mathbf{T}_{a_\perp}\mathbf{e}_2)|_z=\chi(\mathbf{e}_1,\mathbf{e}_2)|_{z+a_\perp}=\chi(\mathbf{e}_1,\mathbf{e}_2)|_z$ where the second identity is a consequence of $\chi(\mathbf{e}_1,\mathbf{e}_2)$ being independent of z.

Let us now suppose that besides satisfying the Bloch-Floquet condition in the xoy plane, $\mathbf{e}_1$ and $\mathbf{e}_2$ have the same property along the z direction. Thus, $\mathbf{e}_1$ is associated with a wave vector of the form $\mathbf{k}_1=\mathbf{k}_\parallel+k_z^{(1)}\hat{\mathbf{u}}_z$, whereas $\mathbf{e}_2$ is associated with $\mathbf{k}_2=\mathbf{k}_\parallel+k_z^{(2)}\hat{\mathbf{u}}_z$; $k_z^{(1)}$ and $k_z^{(2)}$ may be complex valued. It was proven in the previous paragraph that $\chi(\mathbf{T}_{a_\perp}\mathbf{e}_1,\mathbf{T}_{a_\perp}\mathbf{e}_2)=\chi(\mathbf{e}_1,\mathbf{e}_2)$. But for Bloch waves we have that $\mathbf{T}_{a_\perp}\mathbf{e}_1=e^{ik_z^{(1)}a_\perp}\mathbf{e}_1$ and $\mathbf{T}_{a_\perp}\mathbf{e}_2=e^{ik_z^{(2)}a_\perp}\mathbf{e}_2$. Hence, it follows that $(e^{ik_z^{(2)}a_\perp}e^{-ik_z^{(1)*}a_\perp}-1)\chi(\mathbf{e}_1,\mathbf{e}_2)=0$ and thus we have proven that

$$\chi(\mathbf{e}_1,\mathbf{e}_2)=0\quad\text{if }\ k_z^{(2)}-k_z^{(1)*}\neq\frac{2\pi}{a_\perp}l\ \text{ for every }\ l\ \text{ integer}. \tag{42}$$

In particular, the above result implies that if $\mathbf{e}$ is the electric field associated with Bloch-Floquet natural mode associated with a complex wave vector $\mathbf{k}=\mathbf{k}_\parallel+k_z\hat{\mathbf{u}}_z$ [with k_z complex valued, $\text{Im}(k_z)\neq 0$, and $\mathbf{k}_\parallel$ real valued], then $\chi(\mathbf{e},\mathbf{e})=0$, i.e., as could be expected for evanescent modes there is no power flow along the z direction.

We are now ready to determine the macroscopic Poynting vector associated with a superposition of natural modes in a structured material. To this end, let us then consider that the microscopic electric field is the superposition of a set of Bloch-Floquet modes,

$$\mathbf{e}(\mathbf{r})=\sum_n\mathbf{e}_n(\mathbf{r};\mathbf{k}_n) \tag{43}$$

being the wave vector of the form $\mathbf{k}_n=\mathbf{k}_\parallel+k_z^{(n)}\hat{\mathbf{u}}_z$, with $\mathbf{k}_\parallel=(k_x,k_y,0)$ real valued and independent of the considered mode. The z component of the wave vector $k_z^{(n)}$ may be complex valued. It is supposed that $-\pi/a_\perp<\text{Re}\{k_z^{(n)}\}\leq\pi/a_\perp$ (this is always possible due to the assumed Bloch property). As mentioned before, $\chi(\mathbf{e},\mathbf{e})/(-i\omega\mu_0)$ is the power that crosses the transverse cell in the z direction. Hence, since $\chi(\mathbf{e},\mathbf{e})$ is independent of z, it is clear that

$$S_{\mathrm{av},z} = \frac{1}{-i\omega\mu_0}\frac{1}{A_{cell}}\chi(\mathbf{e},\mathbf{e}), \tag{44}$$

where $\mathbf{S}_{\mathrm{av}}$, defined as in Eq. (34), is the cell-averaged microscopic Poynting vector and A_{cell} is the area of the transverse cell Ω_T. Substituting Eq. (43) into Eq. (44) and using formula (42), it is found after straightforward simplifications that

$$S_{\mathrm{av},z} = \sum_{\substack{n \\ k_z^{(n)}\ \mathrm{real}}} \frac{1}{-i\omega\mu_0}\frac{1}{A_{cell}}\chi(\mathbf{e}_n,\mathbf{e}_n) + \sum_{\substack{n,m \\ n\neq m \\ k_z^{(m)}=k_z^{(n)*}}} \frac{1}{-i\omega\mu_0}\frac{1}{A_{cell}}\chi(\mathbf{e}_n,\mathbf{e}_m). \tag{45}$$

To obtain the above formula we have used the fact that the condition "$k_z^{(m)}-k_z^{(n)*}\neq \frac{2\pi}{a_\perp}l$ for every l integer," is equivalent to $k_z^{(m)}\neq k_z^{(n)*}$ since it is assumed that $-\pi/a_\perp<\mathrm{Re}\{k_z\}\leq \pi/a_\perp$. Clearly, the first summation in the right-hand side of Eq. (44) corresponds to the contribution of the propagating modes to the averaged Poynting vector, whereas the second summation yields the contribution of the evanescent modes (note that in general such contribution does not need to vanish because a superposition of evanescent waves may transport power).

Using Eqs. (33) and (44) it is possible to write the first parcel in the right-hand side of Eq. (45) in terms of the macroscopic fields and of the nonlocal dielectric function. On the other hand, it is simple to verify from the definition [Eq. (41)] that

$$\frac{1}{-i\omega\mu_0}\frac{1}{A_{cell}}\chi(\mathbf{e}_n,\mathbf{e}_m) = \frac{1}{V_{cell}}\int_\Omega \mathbf{s}_{m,n}\cdot\hat{\mathbf{u}}_z d^3\mathbf{r} = (\mathbf{s}_{m,n})_{\mathrm{av},z}, \tag{46}$$

where $\mathbf{s}_{m,n}$ is defined consistently with Eq. (15). Hence, using formula (16), we finally conclude that

$$S_{\mathrm{av},z} = \sum_{\substack{n \\ k_z^{(n)}\ \mathrm{real}}} \frac{1}{2}\mathrm{Re}\left\{\left(\mathbf{E}_{\mathrm{av},n}\times\frac{\mathbf{B}^*_{\mathrm{av},n}}{\mu_0}\right)\cdot\hat{\mathbf{u}}_z\right\} - \frac{\omega}{4}\mathbf{E}^*_{\mathrm{av},n}\cdot\frac{\partial\overline{\overline{\varepsilon_{eff}}}}{\partial k_z}(\omega,\mathbf{k}_n)\cdot\mathbf{E}_{\mathrm{av},n} + \sum_{\substack{n,m \\ n\neq m \\ k_z^{(m)}=k_z^{(n)*}}} \frac{1}{4}\left(\mathbf{E}_{\mathrm{av},m}\times\frac{\mathbf{B}^*_{\mathrm{av},n}}{\mu_0}+\mathbf{E}^*_{\mathrm{av},n}\times\frac{\mathbf{B}_{\mathrm{av},m}}{\mu_0}\right)\cdot\hat{\mathbf{u}}_z - \frac{\omega}{4}\mathbf{E}^*_{\mathrm{av},n}\cdot\frac{\partial\overline{\overline{\varepsilon_{eff}}}}{\partial k_z}(\omega,\mathbf{k}_m)\cdot\mathbf{E}_{\mathrm{av},m}. \tag{47}$$

In the above, $\mathbf{E}_{\mathrm{av},n}=\langle\mathbf{e}_n\rangle$ is the amplitude of the macroscopic electric field associated with $\mathbf{e}_n$, defined as in Eq. (7b), $\mathbf{B}_{\mathrm{av},n}=\mathbf{k}_n\times\mathbf{E}_{\mathrm{av},n}/\omega$ is the amplitude of the corresponding macroscopic magnetic induction field, and $\overline{\overline{\varepsilon_{eff}}}(\omega,\mathbf{k})$ is the nonlocal dielectric function of the composite material. The derived formula is mathematically exact, and only requires that the metamaterial is lossless and $\mathbf{k}_\parallel$ is real valued. It establishes the precise relation between the cell-averaged z component of the microscopic Poynting vector and the macroscopic fields obtained after homogenization.

IV. MACROSCOPIC HEATING RATE AND STORED ENERGY

Here, under premises similar to those of Sec. III, we calculate the expressions of the macroscopic stored electromagnetic energy (Sec. IV A) and of the heating rate (Sec. IV B) in an arbitrary structured periodic material with nonmagnetic inclusions.

A. Stored electromagnetic energy

Let us consider a family of electromagnetic natural modes $\mathbf{e}(\mathbf{r};\mathbf{k})$ associated with the frequency $\omega=\omega(\mathbf{k})$ (which determines the dispersion relation of the considered natural modes) in a lossless periodic metamaterial. Thus, for each wave vector $\mathbf{k}$ fixed, $\mathbf{e}(\mathbf{r};\mathbf{k})$ verifies the microscopic Maxwell's equations in Eq. (2) with $\mathbf{j}_e=0$. It should be noted that the considered family of modes is essentially different from that introduced in Sec. III A, even though both families are parameterized by the wave vector $\mathbf{k}$. Indeed, in general, the family of modes considered in Sec. III A is associated with some excitation $\mathbf{j}_e\neq 0$ and is such that the frequency ω is not linked to $\mathbf{k}$.

It is a well known from the theory of dielectric photonic crystals that the cell-averaged microscopic Poynting vector associated with a natural mode with $\mathbf{k}$ real valued is equal to the product of the group velocity $\mathbf{v}_g=\nabla_k\omega$ of the mode and the cell-averaged stored electromagnetic energy [Ref. 31, p. 30] (actually Ref. 31 only considers the case where the inclusions' material is nondispersive, $\frac{\partial\varepsilon}{\partial\omega}=0$, but the result can be readily extended to the dispersive case as shown below),

$$\mathbf{S}_{\mathrm{av}} = W_{\mathrm{av}}\nabla_k\omega, \tag{48a}$$

$$W_{\mathrm{av}} = \frac{1}{4V_{cell}}\int_\Omega \frac{|\mathbf{b}|^2}{\mu_0} d^3\mathbf{r} + \frac{1}{4V_{cell}}\int_\Omega \frac{\partial}{\partial\omega}(\omega\varepsilon)|\mathbf{e}|^2 d^3\mathbf{r}, \tag{48b}$$

where $\mathbf{S}_{\mathrm{av}}$ is defined as in Eq. (34). The first term in the right-hand side of Eq. (48b) is the cell-time-averaged stored magnetic energy, whereas the second term is the cell-time-averaged stored electric energy. Our objective is to relate W_{av} with the macroscopic fields.

To begin with, we note that from Eq. (7) the macroscopic electric field associated with the family of natural modes $\mathbf{e}(\mathbf{r};\mathbf{k})$ is

$$\mathbf{E} = \mathbf{E}(\mathbf{r};\mathbf{k}) = \mathbf{E}_{\mathrm{av}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{49}$$

The macroscopic field verifies the macroscopic Maxwell's equations in Eq. (4) with $\mathbf{J}_e=0$. It can be verified by direct manipulations of Eq. (10) (with $\mathbf{P}_{e,\mathrm{av}}=0$ and $\mathbf{P}_{g,\mathrm{av}} = \overline{\overline{\varepsilon_{eff}}}(\omega,\mathbf{k})\cdot\mathbf{E}_{\mathrm{av}} - \varepsilon_0\mathbf{E}_{\mathrm{av}}$; see Ref. 28, pp. 65–67) that the macroscopic fields are such that

$$\frac{1}{2}\mathrm{Re}\left\{\left(\mathbf{E}_{\mathrm{av}} \times \frac{\mathbf{B}^*_{\mathrm{av}}}{\mu_0}\right)_l\right\} - \frac{\omega}{4}\mathbf{E}^*_{\mathrm{av}} \cdot \frac{\partial\overline{\overline{\varepsilon_{eff}}}}{\partial k_l}(\omega,\mathbf{k})\cdot\mathbf{E}_{\mathrm{av}} = \frac{\partial\omega}{\partial k_l}\left[\frac{1}{4}\frac{|\mathbf{B}_{\mathrm{av}}|^2}{\mu_0} + \frac{1}{4}\mathbf{E}^*_{\mathrm{av}}\cdot\frac{\partial}{\partial\omega}(\omega\overline{\overline{\varepsilon_{eff}}})\cdot\mathbf{E}_{\mathrm{av}}\right] \tag{50}$$

for $l=x,y,z$. But from the fundamental result derived in Sec. III B [Eq. (33)] the left-hand side of the above equation is exactly the lth Cartesian component of the cell-averaged Poynting vector $\mathbf{S}_{\mathrm{av}}$. Therefore, comparing Eqs. (48) and (50), it follows that

$$W_{\mathrm{av}} = \frac{1}{4}\frac{|\mathbf{B}_{\mathrm{av}}|^2}{\mu_0} + \frac{1}{4}\mathbf{E}^*_{\mathrm{av}}\cdot\frac{\partial}{\partial\omega}(\omega\overline{\overline{\varepsilon_{eff}}})\cdot\mathbf{E}_{\mathrm{av}}. \tag{51}$$

It must be emphasized that Eq. (51) is mathematically exact. It assumes that the microscopic electromagnetic field is a Bloch-Floquet natural mode associated with a real-valued wave vector $\mathbf{k}$ and that the metamaterial is lossless. Equation (51) establishes the precise relation between the space-time-averaged stored microscopic electromagnetic energy and the macroscopic effective parameters. The derived formula is completely consistent with the result reported in Ref. 28, which was derived using the macroscopic Maxwell's equations. Here, we have demonstrated that such result is compatible with microscopic theory.

It is interesting to consider the particular case where the material's response is local and is described by local permittivity and permeability tensors, $\overline{\overline{\varepsilon_r}}(\omega)$ and $\overline{\overline{\mu_r}}(\omega)$, so that the nonlocal dielectric function is as in Eq. (1) with $\overline{\overline{\xi}}=\overline{\overline{\zeta}}=0$ (for simplicity, it is assumed that there are no bianisotropic effects). Under such premises, using the vector identities Eqs. (10a) and (36), it is readily found that

$$W_{\mathrm{av}} = \frac{1}{4}\frac{|\mathbf{B}_{\mathrm{av}}|^2}{\mu_0} - \frac{\omega^2}{4\mu_0}\mathbf{B}^*_{\mathrm{av}}\cdot\frac{\partial}{\partial\omega}\left[\frac{1}{\omega}(\overline{\overline{\mu_r}}^{-1} - \overline{\overline{\mathbf{I}}})\right]\cdot\mathbf{B}_{\mathrm{av}} + \frac{1}{4}\mathbf{E}^*_{\mathrm{av}}\cdot\frac{\partial}{\partial\omega}(\omega\varepsilon_0\overline{\overline{\varepsilon_r}})\cdot\mathbf{E}_{\mathrm{av}}. \tag{52}$$

Using now the identity $\frac{\partial}{\partial\omega}\overline{\overline{\mu_r}}^{-1} = -\overline{\overline{\mu_r}}^{-1}\cdot\frac{\partial\overline{\overline{\mu_r}}}{\partial\omega}\cdot\overline{\overline{\mu_r}}^{-1}$, and defining the magnetic field in the classical way, $\mathbf{H}_{\mathrm{av}} = \mu_0^{-1}\overline{\overline{\mu_r}}^{-1}\cdot\mathbf{B}_{\mathrm{av}}$, the stored energy may be rewritten as

$$W_{\mathrm{av}} = \frac{1}{4}\mathbf{H}^*_{\mathrm{av}}\cdot\frac{\partial}{\partial\omega}(\omega\mu_0\overline{\overline{\mu_r}})\cdot\mathbf{H}_{\mathrm{av}} + \frac{1}{4}\mathbf{E}^*_{\mathrm{av}}\cdot\frac{\partial}{\partial\omega}(\omega\varepsilon_0\overline{\overline{\varepsilon_r}})\cdot\mathbf{E}_{\mathrm{av}}, \tag{53}$$

which is coincident with the usual textbook formula for local materials.

B. Heating rate

In this section, we discuss the definition of the macroscopic heating rate in an arbitrary structured material. As in Sec. II, it is supposed that the material is formed by a regular lattice of dielectric or metallic inclusions in a host medium. Here, we admit that the inclusions may be lossy, being characterized by the complex permittivity $\varepsilon=\varepsilon'+i\varepsilon''$.

The microscopic heating rate is

$$q_h(\mathbf{r}) = \frac{1}{2}\mathrm{Re}\{\mathbf{e}(\mathbf{r})\cdot\mathbf{j}^*_d(\mathbf{r})\}, \tag{54}$$

where $\mathbf{e}$ is the microscopic electric field and $\mathbf{j}_d(\mathbf{r})=-i\omega(\varepsilon-\varepsilon_0)\mathbf{e}$ is the induced microscopic density of current. Let us consider an arbitrary solution $(\mathbf{e},\mathbf{b})$ of the homogenization problem formulated in Sec. II B, i.e., a solution of the Maxwell's equations in Eq. (2) for an external current density of the form $\mathbf{j}_e=\mathbf{J}_{e,\mathrm{av}}e^{i\mathbf{k}\cdot\mathbf{r}}$, with $\mathbf{J}_{e,\mathrm{av}}$ a constant vector. As discussed in Sec. II B, for such excitation the microscopic fields have the Bloch-Floquet property. Thus, assuming that the wave vector $\mathbf{k}$ is real valued ($\mathbf{k}$ is determined by the excitation), the spatially averaged heating rate is

$$q_{h,\mathrm{av}} = \frac{1}{2V_{cell}}\int_\Omega \mathrm{Re}\{\mathbf{e}(\mathbf{r})\cdot\mathbf{j}^*_d(\mathbf{r})\}d^3\mathbf{r} = \frac{1}{V_{cell}}\int_\Omega \frac{\omega}{2}\varepsilon''(\mathbf{r})|\mathbf{e}(\mathbf{r})|^2 d^3\mathbf{r}. \tag{55}$$

The objective is to relate the macroscopic heating rate (defined in terms of the microscopic fields as shown above) with the macroscopic fields and with the nonlocal dielectric function.

To this end, we use the fact that an arbitrary solution of the homogenization problem (i.e., the differential system associated with Eq. (2) and $\mathbf{j}_e=\mathbf{J}_{e,\mathrm{av}}e^{i\mathbf{k}\cdot\mathbf{r}}$) has the following integral representation:[20]

$$\mathbf{e}(\mathbf{r}) = \mathbf{E}_{\mathrm{av}}e^{i\mathbf{k}\cdot\mathbf{r}} + i\omega\mu_0\int_\Omega \overline{\overline{G}}_{p0}(\mathbf{r}|\mathbf{r}';\mathbf{k})\cdot\mathbf{j}_d(\mathbf{r}')d^3\mathbf{r}', \tag{56}$$

where $\mathbf{E}_{\mathrm{av}}=\langle\mathbf{e}\rangle$ is the amplitude of the macroscopic field (which depends on the amplitude of the external current

$\mathbf{J}_{e,\mathrm{av}}$) and $\bar{\bar{G}}_{p0}$ is the Green's function dyadic introduced in Ref. 20 that verifies (assuming the time variation $e^{-i\omega t}$),

$$\nabla\times\nabla\times\bar{\bar{G}}_{p0}-\left(\frac{\omega}{c}\right)^2\bar{\bar{G}}_{p0}=\bar{\bar{\mathbf{I}}}e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}\left[\sum_{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}'-\mathbf{r}_{\mathbf{I}})-\frac{1}{V_{cell}}\right] \quad (57)$$

being $\mathbf{r}_{\mathbf{I}}$ a generic lattice point. Thus, substituting Eq. (56) into the definition of the macroscopic heating rate, it is clear that

$$q_{h,\mathrm{av}}=\frac{1}{2V_{cell}}\mathrm{Re}\left\{\mathbf{E}_{\mathrm{av}}\cdot\int_{\Omega}e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{j}_d^*(\mathbf{r})d^3\mathbf{r}\right\}+\frac{1}{2V_{cell}}\mathrm{Re}\left\{i\omega\mu_0\int_{\Omega}\int_{\Omega}\mathbf{j}_d^*(\mathbf{r})\cdot\bar{\bar{G}}_{p0}(\mathbf{r}|\mathbf{r}';\mathbf{k})\cdot\mathbf{j}_d(\mathbf{r}')d^3\mathbf{r}d^3\mathbf{r}'\right\}. \quad (58)$$

However, it is simple to verify that for $\mathbf{k}$ real valued the Green's dyadic $\bar{\bar{G}}_{p0}$ verifies $\bar{\bar{G}}_{p0}(\mathbf{r}'|\mathbf{r};\mathbf{k})=[\bar{\bar{G}}_{p0}(\mathbf{r}|\mathbf{r}';\mathbf{k})]^\dagger$ (the superscript $\dagger$ represents the conjugate transpose) and this implies that the second term in the right-hand side of Eq. (58) vanishes. On the other hand, the integral associated with the first term can be easily related to the generalized polarization vector $\mathbf{P}_{g,\mathrm{av}}$ through the use of Eq. (9). Thus, we conclude that

$$q_{h,\mathrm{av}}=\frac{1}{2}\mathrm{Re}\{i\omega\mathbf{E}_{\mathrm{av}}\cdot\mathbf{P}_{g,\mathrm{av}}^*\}=\frac{1}{2}\mathrm{Re}\{i\omega\mathbf{E}_{\mathrm{av}}\cdot\mathbf{D}_{g,\mathrm{av}}^*\}, \quad (59)$$

where the second identity is a consequence of the constitutive relation (8). Therefore we have demonstrated that, provided the microscopic fields are Bloch-Floquet waves associated with a real-valued wave vector $\mathbf{k}$ and the external excitation is of the form $\mathbf{j}_e=\mathbf{J}_{e,\mathrm{av}}e^{i\mathbf{k}\cdot\mathbf{r}}$ with $\mathbf{J}_{e,\mathrm{av}}$ a constant vector, then the spatial averaged heating rate is *exactly* $q_{h,\mathrm{av}}=\frac{1}{2}\mathrm{Re}\{-i\omega\mathbf{E}_{\mathrm{av}}^*\cdot\bar{\bar{\varepsilon}}_{eff}(\omega,\mathbf{k})\cdot\mathbf{E}_{\mathrm{av}}\}$. It should be mentioned that considering $\mathbf{k}$ real valued is not incompatible with the presence of loss because as mentioned before such spatial variation can be imposed by an external excitation of the form $\mathbf{j}_e=\mathbf{J}_{e,\mathrm{av}}e^{i\mathbf{k}\cdot\mathbf{r}}$. Thus, in presence of loss, Eq. (59) does not apply to the natural Bloch-Floquet modes of the material (i.e., Bloch-Floquet waves associated with $\mathbf{j}_e=0$). Indeed, at least for a general spatially dispersive material, it is not possible to obtain a strict relation between the averaged heating rate and the macroscopic fields when the material is lossy and $\mathbf{k}$ is complex valued.

Let us consider now the particular case where the material's response is local (with no magnetoelectric coupling) and is described by local permittivity and permeability tensors, $\bar{\bar{\varepsilon}}_r(\omega)$ and $\bar{\bar{\mu}}_r(\omega)$, as also considered in the end of Sec. IV A. In such conditions, the macroscopic heating rate is given by

$$q_{h,\mathrm{av}}=\frac{1}{2}\mathrm{Re}\{-i\omega\mathbf{E}_{\mathrm{av}}^*\cdot\varepsilon_0\bar{\bar{\varepsilon}}_r(\omega)\cdot\mathbf{E}_{\mathrm{av}}\}+\frac{1}{2}\mathrm{Re}\left\{-i\omega\mathbf{E}_{\mathrm{av}}^*\cdot\left[\frac{1}{\omega^2\mu_0}\mathbf{k}\times(\bar{\bar{\mu}}_r^{-1}-\bar{\bar{\mathbf{I}}})\times\mathbf{k}\right]\cdot\mathbf{E}_{\mathrm{av}}\right\}. \quad (60)$$

Using now the vector properties Eqs. (10a) and (36), it is readily found that

$$q_{h,\mathrm{av}}=\frac{1}{2}\mathrm{Re}\{-i\omega[\mathbf{E}_{\mathrm{av}}^*\cdot\varepsilon_0\bar{\bar{\varepsilon}}_r(\omega)\cdot\mathbf{E}_{\mathrm{av}}-\mathbf{B}_{\mathrm{av}}^*\cdot\mu_0^{-1}\bar{\bar{\mu}}_r^{-1}\cdot\mathbf{B}_{\mathrm{av}}]\}. \quad (61)$$

Defining the macroscopic magnetic field in the usual way, $\mathbf{H}_{\mathrm{av}}=\mu_0^{-1}\bar{\bar{\mu}}_r^{-1}\cdot\mathbf{B}_{\mathrm{av}}$ we may rewrite the above formula as

$$q_{h,\mathrm{av}}=\frac{1}{2}\mathrm{Re}\{-i\omega[\mathbf{E}_{\mathrm{av}}^*\cdot\varepsilon_0\bar{\bar{\varepsilon}}_r(\omega)\cdot\mathbf{E}_{\mathrm{av}}+\mathbf{H}_{\mathrm{av}}^*\cdot\mu_0\bar{\bar{\mu}}_r(\omega)\cdot\mathbf{H}_{\mathrm{av}}]\} \quad (62)$$

consistently with a classical textbook formula.[27] Hence, we have shown that the traditional formula for the macroscopic heating rate in local media is completely consistent with the results obtained directly from microscopic theory.

It is interesting to note that Eq. (62) is valid for arbitrary $\mathbf{E}_{\mathrm{av}}$ and arbitrary (real valued) $\mathbf{k}$ since both these parameters are determined by the external excitation. On the other hand, from Eq. (55) it is clear that for passive materials $q_{h,\mathrm{av}}\geq 0$. But since $\mathbf{E}_{\mathrm{av}}$ and $\mathbf{k}$ are independent parameters determined by the source, and because $\bar{\bar{\varepsilon}}_r(\omega)$ and $\bar{\bar{\mu}}_r(\omega)$ are symmetric tensors, this is only possible if both $\mathrm{Im}\{\bar{\bar{\varepsilon}}_r(\omega)\}$ and $\mathrm{Im}\{\bar{\bar{\mu}}_r\}$ are nonnegative tensors ($\bar{\bar{\mathbf{A}}}$ is a non-negative tensor if $\mathbf{v}^*\cdot\bar{\bar{\mathbf{A}}}\cdot\mathbf{v}\geq 0$ for arbitrary $\mathbf{v}$). Therefore in a local isotropic material the imaginary part of the permeability cannot be negative, unlike what was erroneously claimed in Refs. 17 and 32. Such conclusion was drawn in Ref. 32 based on numerical calculations of the effective parameters of a metamaterial. However, as argued by Efros in Ref. 33, such result would contradict the second law of thermodynamics and can only be explained by the periodicity of the material and the emergence of nonlocal effects, an explanation with which the authors of Ref. 32 also agreed.[34]

V. CONCLUSION

In this work we have shown that in some circumstances it is possible to establish a rigorous mathematical relation between the macroscopic fields and macroscopic quadratic physical entities such as the cell-averaged microscopic Poynting vector, the cell-averaged microscopic heating rate, and the cell-averaged stored microscopic electromagnetic energy. The derived formulas (which agree with well-known textbook formulas when these are available) in general require that the microscopic field is a Bloch-Floquet natural mode of the structured metallic-dielectric periodic material (or more generally a superposition of such modes). With the exception of the formula of the heating rate [Eq. (59)], all the results require that the metamaterial is lossless. In fact, apparently, for an absorbing spatially dispersive medium it is not possible to relate the macroscopic Poynting vector (or stored energy) with the macroscopic fields and with $\overline{\overline{\varepsilon_{eff}}}(\omega,\mathbf{k})$, even if we try to define these entities directly from the macroscopic Maxwell's equations as done in Ref. 28. As discussed in Ref. 28, p. 63, the reason is that the nonlocal dielectric function determines the response of the system to the macroscopic electric field and that in case of loss this response may be the same also in conditions where the energy stored in the system is different (i.e., in case of loss we can have two linear systems with the same response but with different stored energies). Therefore in case of an absorbing medium the nonlocal dielectric function $\overline{\overline{\varepsilon_{eff}}}(\omega,\mathbf{k})$ does not convey sufficient information to retrieve the stored energy and the Poynting vector.

In particular, we have shown that the traditional definitions of the Poynting vector and heating rate in local media are absolutely accurate. The recent claims[14] that the correct definition for the Poynting vector in metamaterials that exhibit artificial magnetism should be $\mathbf{S}_{av}=\frac{1}{2}\mathrm{Re}\{\mathbf{E}\times\frac{\mathbf{B}^*}{\mu_0}\}$ were found completely unsubstantiated.

One of original results of this work [Eq. (47)] establishes the exact relation between the macroscopic Poynting vector and the macroscopic fields for a superposition of Bloch-Floquet natural modes, possibly associated with a complex wave vector. This result may be quite useful for study of the problem of plane-wave incidence at an interface between different media and, in particular, to define boundary conditions consistent with the conservation of the power flow in spatially dispersive media.[35–38] These topics will be addressed in future work.

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*Author to whom correspondence should be addressed; mario.silveirinha@co.it.pt

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